MATHEMATICS SPECIALIST 3CD

SEMESTER 1 2010

TEST 1

Calculator Free

Reading Time: 5 minutes
Working Time: 70 minutes

Total Marks: 51

1. [1, 2, 3 marks]

Spaceman Spiff takes off from home base, coordinates (0, 0, 0) and two minutes later is at position (6, 3, 2) km. His radio tells him that an ugly Xgrty beast has been sighted by two spotter craft, Alpha at A(14, 7, 3) km and Beta at B(26, 34, 9) km and such that $\vec{AX}:\vec{XB} = 2:1$.

(a) How far did Spaceman Spiff fly in the first two minutes?

$d = \sqrt{6^2+3^2+2^2}$

$= \sqrt{49}$

$= 7$ km ✓

(b) Determine the coordinates of the ugly Xgrty beast from home base.

$\vec{AB} = \begin{pmatrix}12\\27\\6\end{pmatrix}$

$\vec{AX} = \frac{2}{3}\vec{AB}$

$= \begin{pmatrix}8\\18\\4\end{pmatrix}$

$\vec{OX} = \vec{OA} + \vec{AX}$

$= \begin{pmatrix}14\\7\\3\end{pmatrix} + \begin{pmatrix}8\\18\\4\end{pmatrix}$

$= \begin{pmatrix}22\\25\\7\end{pmatrix}$ $\therefore$ (22, 25, 7)

A —2— X —1— B

WORKING ✓
ANS ✓
(−½ mk if not coord)

Spaceman Spiff then alters his velocity to a vector of (3, 4, 1) km/min.

(c) Through what angle did Spaceman Spiff's velocity change? Give your answer as an exact trigonometric function.

$\begin{pmatrix}6\\3\\2\end{pmatrix}\cdot\begin{pmatrix}3\\4\\1\end{pmatrix} = 7\sqrt{9+16+1}\cos\theta$

$18+12+2 = 7\sqrt{26}\cos\theta$

$\frac{32}{7\sqrt{26}} = \cos\theta$

using dot product ✓
correct values ✓
$\left.\begin{matrix}\cos\theta = \\ \theta = \cos^{-1} =\end{matrix}\right\}$ ✓

2. [2, 3 marks]

(a) Evaluate, showing your working, the exact value of $\lim_{h \to 0}\left(\frac{\sin(\frac{\pi}{3}+h)-\sin(\frac{\pi}{3})}{h}\right)$ ✓

This is $\left.\frac{d(\sin x)}{dx}\right|_{x=\frac{\pi}{3}} = \cos x \ @\ x=\frac{\pi}{3}$

$= \cos\frac{\pi}{3}$

$= \frac{1}{2}$ ✓

Appropriate Reasoning ✓

Answer ✓

(b) Prove algebraically the limit $\lim_{x \to 0}\frac{1-\cos x}{x} = 0$

$\lim_{x\to 0}\frac{1-\cos x}{x} \times \frac{1+\cos x}{1+\cos x}$ ✓ $= \lim_{x \to 0}\frac{1-\cos^2 x}{x(1+\cos x)}$

$= \lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}$ ✓

$= \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin x}{1+\cos x}$ ✓

$= 1 \cdot \frac{0}{2}$

$= 0$

3. [2, 3, 3 marks]

(a) Given that $y = 2\sin\theta\cos\theta$, determine $\frac{dy}{d\theta}$ $= 2\cos\theta\cos\theta - 2\sin\theta\sin\theta$ ✓

$= 2(\cos^2\theta - \sin^2\theta)$ ✓

$= 2\cos 2\theta$

(does not need to be Factorised)

$y = \sin 2\theta$ ✓

$\frac{dy}{d\theta} = 2\cos 2\theta$ ✓

(b) Determine $\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{3}}$ given that $y = 5\cos^3 2x$

$\frac{dy}{dx} = 30\cos^2 2x \cdot {}^{-}\sin 2x$

$= {}^{-}30\cos^2 2x\sin 2x$ ✓

$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{3}} = {}^{-}30\cos^2\left(\frac{2\pi}{3}\right)\sin\left(\frac{2\pi}{3}\right)$

$= {}^{-}30\left(-\frac{1}{2}\right)^2 \cdot \left(\frac{\sqrt{3}}{2}\right)$ ✓

$= \frac{-30\sqrt{3}}{8}$

$= \frac{-15\sqrt{3}}{4}$ ✓

(c) Determine the equation of the tangent to the curve y = 3tan x at $(\frac{\pi}{4}, 3)$.

$$\frac{dy}{dx} = \frac{3}{\cos^2 x} \quad ✓$$

$$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{4}} = \frac{3}{\cos^2(\frac{\pi}{4})}$$

$$= \frac{3}{(\frac{1}{\sqrt{2}})^2}$$

$$= 6 \quad ✓$$

$$y = 6x + c$$

$$3 = 6(\tfrac{\pi}{4}) + c$$

$$= \frac{3\pi}{2} + c$$

$$c = 3 - \frac{3\pi}{2}$$

$$\therefore y = 6x + (3 - \tfrac{3\pi}{2}) \quad ✓$$

$$= 6x + \frac{(6 - 3\pi)}{2}$$

4. [1, 2 marks]

a, **b**, and **c** are vectors such that **a** = 3**i** + 5**j** – 4**k**, **b** = 6**i** – 2**j** + 2**k**, and **c** = 5**i** + **j** + 5**k**.

(a) Show that **a** and **b** are perpendicular.

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = (3\underset{\sim}{i} + 5\underset{\sim}{j} - 4\underset{\sim}{k}) \cdot (6\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}) \quad ✓$$

$$= 18 - 10 - 8$$

$$= 0$$

$$\therefore \underset{\sim}{a} \perp \underset{\sim}{b}$$

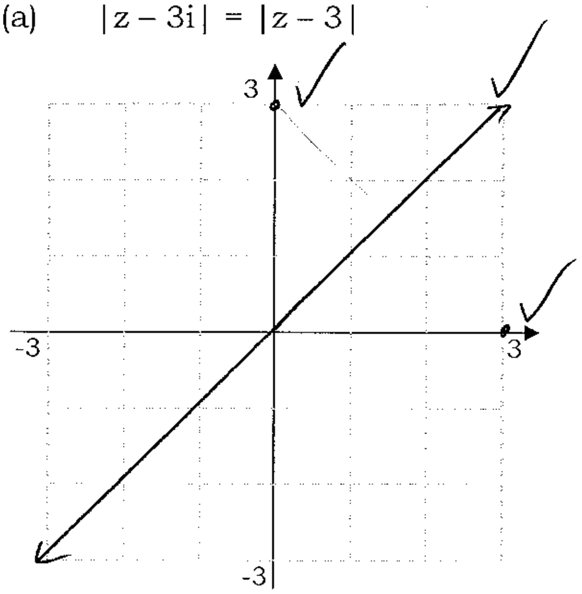
(b) Vectors **a** and **c** are also perpendicular, yet vectors **b** and **c** are <u>not</u> parallel. Explain (with the aid of diagrams if needed) how this can occur.

Any reasonable argument ✓✓

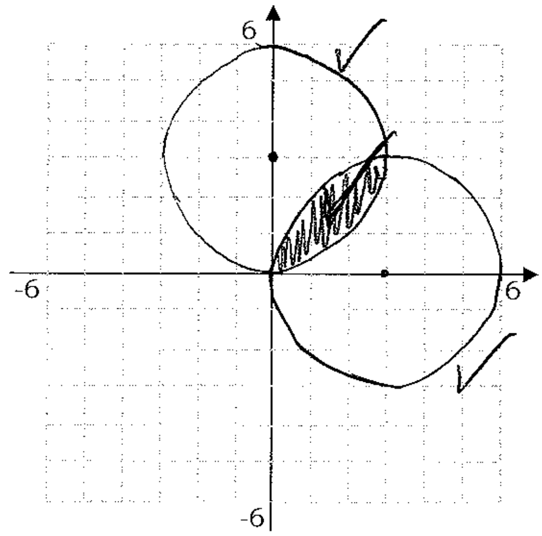
5. [3, 3 marks]

Sketch the set of points z, in the complex plane, that satisfy the following regions:

(a) $|z - 3i| = |z - 3|$



(b) $\{z: |z - 3i| \le 3\} \cap \{z: |z - 3| \le 3\}$



6. [2, 1, 2, 2, 3 marks]

(a) Determine the exact distance between the points A[3, $\frac{\pi}{6}$] and B[4, $\frac{\pi}{2}$].

$d^2 = 3^2 + 4^2 - 2(3)(4)\cos\frac{\pi}{3}$

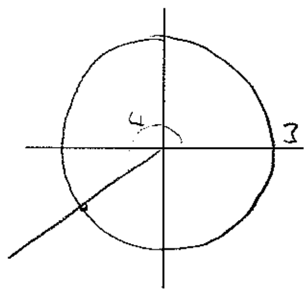
$= 25 - 24(\frac{1}{2})$

$= 13$

$d = \sqrt{13}$

(b) Calculate the point(s) of intersection of the curves r = 3 and θ = 4, for r ≥ 0 and 0 ≤ θ ≤ 2π. Give your answer in exact polar form.

[3, 4]

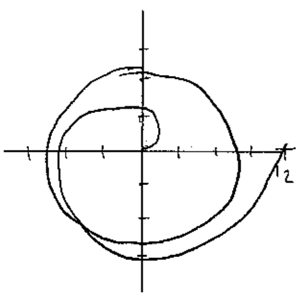


(c) Calculate the point(s) of intersection of the curves r = 2θ and r = 7, for r ≥ 0 and 0 ≤ θ ≤ 2π. Give your answer in exact polar form.

$7 = 2\theta$

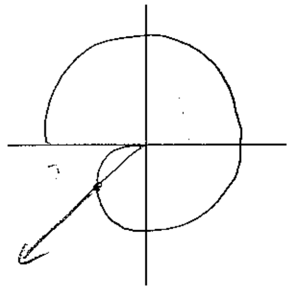
$\theta = 3.5$

[7, 3.5]



(d) Calculate the point(s) of intersection of the curves $\{r: r = -\theta \text{ , } 0 \le \theta \le 2\pi\}$ and $\{\theta: \theta = \frac{4\pi}{3} \text{ , } r \ge 0\}$. Give your answer in exact polar form.

$\left[\frac{\pi}{3}, \frac{4\pi}{3}\right]$ or $[0,0]$



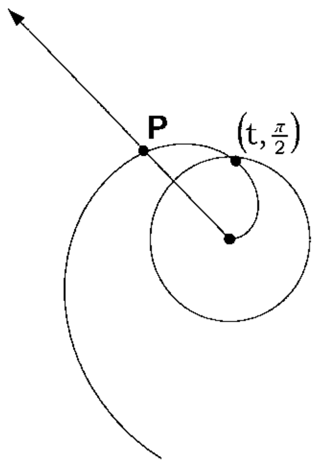
(e) The diagram below shows the polar curves $r = k\theta$, $\theta = m$ and $r = t$ (where k, m and t are constants).

If point P has polar coordinates $\left[\frac{3\pi}{2}, \frac{3\pi}{4}\right]$, determine, **exactly**, the values of k, m and t.

$m = \frac{3\pi}{4}$

$k = 2$

$t = \pi$



7. [1, 5, 1, 1, 1 marks]

(a) If $z = \text{cis } \frac{\pi}{4}$ and $w = \text{cis } \frac{\pi}{6}$

$$= \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

(i) Express $\frac{z}{w}$ in polar form.

$\frac{z}{w} = \text{cis}\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

$= \text{cis}\left(\frac{\pi}{12}\right)$

(ii) Express z and $\frac{z}{w}$ in Cartesian form and give $\frac{z}{w}$ with a rationalised denominator.

$z = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

$= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$\frac{z}{w} = \frac{\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}}{\frac{\sqrt{3}}{2} + i\frac{1}{2}}$

$= \frac{\sqrt{2} + i\sqrt{2}}{\sqrt{3} + i}$

$= \frac{\sqrt{2} + i\sqrt{2}}{\sqrt{3} + i} \times \frac{\sqrt{3} - i}{\sqrt{3} - i}$

$= \frac{\sqrt{6} - i\sqrt{2} + i\sqrt{6} + \sqrt{2}}{3+1}$

$= \frac{(\sqrt{6}+\sqrt{2}) + i(\sqrt{6}-\sqrt{2})}{4}$

(iii) Use the results above to give the exact value for $\sin \frac{\pi}{12}$

$$\text{cis}\left(\frac{\pi}{12}\right) = \frac{(\sqrt{6}+\sqrt{2}) + i(\sqrt{6}-\sqrt{2})}{4}$$

Equating imaginary parts: $\therefore \sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$ ✓

(b) If $z = 3 \text{ cis } \frac{\pi}{4}$ and $w = 2 \text{ cis } \frac{\pi}{3}$, express the following in polar form

(i) iw $\quad i = \text{cis}\left(\frac{\pi}{2}\right)$ $\quad iw = 2\text{cis}\frac{5\pi}{6}$ ✓

(ii) $\left(\frac{w}{z}\right)^3$

$$\frac{w}{z} = \frac{2}{3}\text{cis}\frac{\pi}{12}$$

$$\left(\frac{w}{z}\right)^3 = \left(\frac{2}{3}\text{cis}\frac{\pi}{12}\right)^3$$

$$= \frac{8}{27}\text{cis}\frac{3\pi}{12}$$

$$= \frac{8}{27}\text{cis}\frac{\pi}{4}$$ ✓

8. [4 marks]

Solve the following equation in polar form:

$-\pi < \theta \leq \pi$

$z^6 + 64 = 0$ where $z = r \operatorname{cis} \theta$ with r =0, -p< ?=p

$z^6 = -64$

$= 64 \operatorname{cis} \pi$ ✓

$z = (64 \operatorname{cis} (\pi + 2k))^{1/6}$

$= 64^{1/6} \operatorname{cis} \frac{1}{6}(\pi + 2k\pi)$

$= 2 \operatorname{cis} \left(\frac{\pi}{6} + \frac{2k\pi}{6}\right)$ ✓

$k=0 \quad z = 2\operatorname{cis}\frac{\pi}{6}$

$k=1 \quad z = 2\operatorname{cis}\frac{3\pi}{6}$

$= 2\operatorname{cis}\frac{\pi}{2}$

$k=2 \quad z = 2\operatorname{cis}\frac{5\pi}{6}$ ✓

$k=3 \quad z = 2\operatorname{cis}\frac{7\pi}{6}$

$= 2\operatorname{cis}\left(\frac{-5\pi}{6}\right)$ ✓

$k=4 \quad z = 2\operatorname{cis}\left(\frac{9\pi}{6}\right)$

$= 2\operatorname{cis}\frac{-\pi}{2}$

$k=5 \quad z = 2\operatorname{cis}\frac{11\pi}{6}$

$= 2\operatorname{cis}\left(\frac{-\pi}{6}\right)$

If $0 \leq \theta \leq 2\pi$ then $-\frac{1}{2}$mk